shooting method. This solution method appears to be computationally intense for systems in which the controls experience many switches; with all controls saturating, example 2 requires solving for 89 unknown parameters. We have found, however, that, by sweeping the parameter p in a homotopylike fashion, beginning with the condition that none of the controls saturates, insight to the optimal switching structure for the formal minimum-time problem may be gained. The problem formulation, numerical solution procedure, and example problems are covered in greater detail in Chap. 2 of Ref. 10.

References

[1]Bryson, A. E., and Ho, Y.-C., *Applied Optimal Control*, Blaisdell, Waltham, MA, 1969, Chap. 3.

[2]Junkins, J. L., and Turner, J. D., *Optimal Spacecraft Rotational Maneuvers*, Vol. 3, Studies in Astronautics, Elsevier Scientific, New York, 1985, Chaps. 6 and 7.

[3]Junkins, J. L., Rahman, Z., and Bang, H., "Near-Minimum-Time Control of Distributed Parameter Systems: Analytical and Experimental Results," *Journal of Guidance, Control, and Dynamics*, Vol. 14, No. 2, 1991, pp. 406–415.

[4]Aspinwall, D. W., "Acceleration Profiles for Minimizing Residual Response," *Journal of Dynamic Systems, Measurement and Control*, Vol. 102, No. 1, 1980, pp. 3–6.

[5]Swigert, C. J., "Shaped Torque Techniques," *Journal of Guidance and Control*, Vol. 3, No. 5, 1980, pp. 460–467.

[6]Hestenes, M. R., *Calculus of Variations and Optimal Control Theory*, Applied Mathematics Series, Wiley, New York, 1966, Chap. 8.

[7]Bryson, A. E., Denham, W. F., and Dreyfus, S. E., "Optimal Programming Problems with Inequality Constraints I: Necessary Conditions for Extremal Solutions," *AIAA Journal*, Vol. 1, No. 11, 1963, pp. 2544–2550.

[8]Stoer, J., and Bulirsch, R., *Introduction to Numerical Analysis*, Springer-Verlag, New York, 1980.

[9]Bilimoria, K. D., and Wie, B., "Time-Optimal Three-Axis Reorientation of a Rigid Spacecraft," *Journal of Guidance, Control, and Dynamics*, Vol. 16, No. 3, 1993, pp. 446–452.

[10]Hurtado, J. E., "Some New Methods for Optimal Control of Constrained Dynamical Systems," Ph.D. Thesis, Dept. of Aerospace Engineering, Texas A&M Univ., College Station, TX, Aug. 1995.

Time-Variant Receding-Horizon Control of Nonlinear Systems

Toshiyuki Ohtsuka*

University of Tsukuba, Tsukuba, Ibaraki 305, Japan

Introduction

THE objective of this Note is to extend the conventional real-time optimization algorithm[1] to the receding-horizon state feedback control problem with explicit time-variant parameters. An algorithm is derived in a modified manner, which results in less computation and less data storage than a direct extension of the conventional algorithm. A tracking control problem of a two-wheeled car is employed as a numerical example. A simulation result demonstrates closed-loop characteristics of the designed tracking control law.

Problem Formulation

The dynamic system treated here is expressed in the following differential equation:

$$\frac{\mathrm{d}\boldsymbol{x}(t)}{\mathrm{d}t} = \boldsymbol{f}[\boldsymbol{x}(t), \boldsymbol{u}(t), \boldsymbol{p}(t)] \tag{1}$$

where $\boldsymbol{x}(t) \in \boldsymbol{R}^n$ denotes the state vector, $\boldsymbol{u}(t) \in \boldsymbol{R}^m$ the control input vector, and $\boldsymbol{p}(t) \in \boldsymbol{R}^r$ the vector of time-variant parameters. The time-variant parameter $\boldsymbol{p}(t)$ is assumed to be known. An optimal state feedback law is designed so as to minimize a receding-horizon performance index:

$$J = \varphi[\boldsymbol{x}(t+T), \boldsymbol{p}(t+T)] + \int_t^{t+T} L[\boldsymbol{x}(t'), \boldsymbol{u}(t'), \boldsymbol{p}(t')]\,\mathrm{d}t' \tag{2}$$

Because the time-variant parameter $\boldsymbol{p}(t)$ is included both in the state equation and in the performance index, the present problem can deal with not only control of time-variant systems but also tracking control problems. A command input is regarded as a time-variant parameter in a tracking control problem. The performance index evaluates the performance from the present time t to the finite future $t + T$. Because the time interval in the performance index is finite, the integrand in the performance index does not have to converge to zero as time increases. Therefore, the receding-horizon tracking control law can be determined even if the tracking error does not converge to zero with any control input. That is, any feasibility conditions on $\boldsymbol{p}(t)$ and any controllability conditions on the system are not necessary, although asymptotic tracking is not guaranteed, in general, by the present formulation of the receding-horizon tracking control. Some analytical results are found in Ref. 2 about the receding-horizon tracking control problem under restrictive conditions.

The performance index equation (2) is minimized for each time t starting from $\boldsymbol{x}(t)$. By denoting the trajectory $\boldsymbol{x}(t+\tau)$ starting from $\boldsymbol{x}(t)$ as $\boldsymbol{x}^*(\tau, t)$, the present receding-horizon control problem can be converted to a family of finite horizon optimal control problems on the τ axis parameterized by time t as follows.

Minimize:

$$J = \varphi[\boldsymbol{x}^*(T, t), \boldsymbol{p}(t+T)] + \int_0^T L[\boldsymbol{x}^*(\tau, t), \boldsymbol{u}^*(\tau, t), \boldsymbol{p}(t+\tau)]\,\mathrm{d}\tau \tag{3}$$

subject to:

$$\boldsymbol{x}^*_\tau(\tau, t) = \boldsymbol{f}[\boldsymbol{x}^*(\tau, t), \boldsymbol{u}^*(\tau, t), \boldsymbol{p}(t+\tau)] \tag{4}$$

with the initial state on the τ axis $\boldsymbol{x}^*(0, t)$ given by

$$\boldsymbol{x}^*(0, t) = \boldsymbol{x}(t) \tag{5}$$

The actual control input $\boldsymbol{u}(t)$ is given by

$$\boldsymbol{u}(t) = \boldsymbol{u}^*(0, t) \tag{6}$$

where the asterisk denotes a variable in the converted optimal control problem so as to distinguish it from its correspondence in the original problem. Note that the converted optimal control problem is a standard Bolza-type problem on the τ axis for a pair of fixed t and T and the actual control input is given by the initial value of the optimal control input minimizing the performance index. The first-order necessary conditions for the optimal solution are readily obtained as a two-point boundary-value problem (TPBVP) by the calculus of variations as[3]

$$\boldsymbol{x}^*_\tau(\tau, t) = H_\lambda^T, \qquad \boldsymbol{x}^*(0, t) = \boldsymbol{x}(t) \tag{7}$$

$$\boldsymbol{\lambda}^*_\tau(\tau, t) = -H_x^T, \qquad \boldsymbol{\lambda}^*(T, t) = \varphi_x^T[\boldsymbol{x}^*(T, t), \boldsymbol{p}(t+T)] \tag{8}$$

$$H_u = 0 \tag{9}$$

where $\boldsymbol{\lambda}^*(\tau, t) \in \boldsymbol{R}^n$ denotes the costate, H denotes the Hamiltonian defined as

$$H = L + \boldsymbol{\lambda}^{*T}\boldsymbol{f} \tag{10}$$

and H_x denotes the partial derivative of H with respect to $\boldsymbol{x}^*$, and so on.

Because the state and costate at $\tau = T$ are determined by the Euler–Lagrange equations (7–9) from the state and costate at $\tau = 0$, the TPBVP can be regarded as a nonlinear algebraic equation with respect to the costate at $\tau = 0$ as

$$\boldsymbol{F}[\boldsymbol{\lambda}(t), \boldsymbol{x}(t), T, t] = \boldsymbol{\lambda}^*(T, t) - \varphi_x^T[\boldsymbol{x}^*(T, t), \boldsymbol{p}(t+T)] = 0 \tag{11}$$

Received Nov. 14, 1996; revision received Sept. 15, 1997; accepted for publication Sept. 16, 1997.

*Assistant Professor, Institute of Engineering Mechanics, 1-1-1 Tennodai. E-mail: ohtsuka@kz.tsukuba.ac.jp. Member AIAA.

where $\boldsymbol{\lambda}(t)$ denotes the costate at $\tau = 0$, $\boldsymbol{\lambda}^*(0, t)$. The unknown costate $\boldsymbol{\lambda}(t)$ is determined implicitly by Eq. (11). The actual control input is given by

$$\boldsymbol{u}(t) = \arg\{H_u[\boldsymbol{x}(t), \boldsymbol{\lambda}(t), \boldsymbol{u}(t), \boldsymbol{p}(t)] = 0\} \tag{12}$$

If T is chosen as a smooth function of t, then the solution of the equation $\boldsymbol{F}[\boldsymbol{\lambda}(t), \boldsymbol{x}(t), T(t), t] = 0$ can be tracked with respect to time t. In the next section, a differential equation of the unknown costate $\boldsymbol{\lambda}(t)$ is derived so that Eq. (11) is satisfied.

Feedback Control Algorithm

Because the nonlinear equation (11) has to be satisfied at any time t, $\mathrm{d}\boldsymbol{F}[\boldsymbol{\lambda}(t), \boldsymbol{x}(t), T(t), t]/\mathrm{d}t = 0$ holds along the trajectory of the closed-loop system of the receding-horizon control. To avoid accumulation of numerical error, this Note employs the following condition:

$$\frac{\mathrm{d}\boldsymbol{F}}{\mathrm{d}t} = A_s \boldsymbol{F} \tag{13}$$

where A_s denotes a stable matrix. Equation (13) results in the exponential attenuation of the error in $\boldsymbol{F}$. An ordinary differential equation of the unknown costate $\boldsymbol{\lambda}(t)$ is derived from the condition equation (13). Furthermore, to start from a trivial solution of the TPBVP with $T = 0$, T is regarded as a smooth function of the time t such that $T(0) = 0$ and $T(t) \to T_f$ $(t \to \infty)$, where T_f is prescribed by a designer. The initial condition for the differential equation of the costate is given by

$$\boldsymbol{\lambda}(0) = \varphi_x^T[\boldsymbol{x}(0), \boldsymbol{p}(0)] \tag{14}$$

which is a trivial solution of the TPBVP for $T(0) = 0$. Although the optimal control problem itself is meaningless with $T = 0$, and any control can be used at $t = 0$, Eq. (14) is employed to prevent a large jump in control.

To evaluate the derivative of $\boldsymbol{\lambda}(t) = \boldsymbol{\lambda}^*(0, t)$ with respect to time, we consider the partial derivatives of the optimal trajectory. Although an algorithm similar to one in Ref. 1 can also be obtained for time-variant problems,[4] this Note derives an algorithm of a different form, which is more suitable for time-variant problems. The main difference is that $\boldsymbol{x}_t^* - \boldsymbol{x}_\tau^*$ and $\boldsymbol{\lambda}_t^* - \boldsymbol{\lambda}_\tau^*$ are used instead of $\boldsymbol{x}_t^*$ and $\boldsymbol{\lambda}_t^*$. Partial differentiation of Eqs. (7–9) with respect to time t and τ yields the following linear differential equation:

$$\frac{\partial}{\partial \tau}\begin{bmatrix} \boldsymbol{x}_t^* - \boldsymbol{x}_\tau^* \\ \boldsymbol{\lambda}_t^* - \boldsymbol{\lambda}_\tau^* \end{bmatrix} = \begin{bmatrix} A & -B \\ -C & -A^T \end{bmatrix}\begin{bmatrix} \boldsymbol{x}_t^* - \boldsymbol{x}_\tau^* \\ \boldsymbol{\lambda}_t^* - \boldsymbol{\lambda}_\tau^* \end{bmatrix} \tag{15}$$

where

$$A = \boldsymbol{f}_x - \boldsymbol{f}_u H_{uu}^{-1} H_{ux}, \qquad B = \boldsymbol{f}_u H_{uu}^{-1} \boldsymbol{f}_u^T$$
$$C = H_{xx} - H_{xu} H_{uu}^{-1} H_{ux} \tag{16}$$

Note that $\boldsymbol{p}_t(t+\tau) = \boldsymbol{p}_\tau(t+\tau)$ holds and $\boldsymbol{p}_t$ and $\boldsymbol{p}_\tau$ are canceled in Eq. (15). Therefore, $\boldsymbol{p}_t(t+\tau)$ does not have to be stored for $0 \le \tau \le T$. The derivative of the function $\boldsymbol{F}$ with respect to time is expressed as

$$\frac{\mathrm{d}\boldsymbol{F}}{\mathrm{d}t} = \boldsymbol{\lambda}_t^*(T, t) - \varphi_{xx}\boldsymbol{x}_t^*(T, t) - \varphi_{xp}\boldsymbol{p}_t(t+T)$$
$$+ \left[\boldsymbol{\lambda}_\tau^*(T, t) - \varphi_{xx}\boldsymbol{x}_\tau^*(T, t) - \varphi_{xp}\boldsymbol{p}_\tau(t+T)\right]\frac{\mathrm{d}T}{\mathrm{d}t} \tag{17}$$

where the derivatives $\boldsymbol{x}_\tau^*(T, t)$ and $\boldsymbol{\lambda}_\tau^*(T, t)$ are given by the Euler–Lagrange equations (7–9).

The relationship between the costate and other variables is assumed in this case as

$$\boldsymbol{\lambda}_t^* - \boldsymbol{\lambda}_\tau^* = S(\tau, t)\left(\boldsymbol{x}_t^* - \boldsymbol{x}_\tau^*\right) + \boldsymbol{c}(\tau, t) \tag{18}$$

which is different from the one used in Ref. 1. It is apparent from Eqs. (13), (17), and (18) that the following equations have to hold:

$$S(T, t) = \varphi_{xx}|_{\tau=T} \tag{19}$$

$$\boldsymbol{c}(T, t) = \left(H_x^T + \varphi_{xx}\boldsymbol{f} + \varphi_{xp}\boldsymbol{p}_t\right)\Big|_{\tau=T}\left(1 + \frac{\mathrm{d}T}{\mathrm{d}t}\right) + A_s\boldsymbol{F} \tag{20}$$

Furthermore, Eqs. (15) and (18) imply that the matrix $S(\tau, t) \in \boldsymbol{R}^{n\times n}$ and the vector $\boldsymbol{c}(\tau, t) \in \boldsymbol{R}^n$ have to satisfy the following differential equations:

$$S_\tau = -A^T S - SA + SBS - C \tag{21}$$

$$\boldsymbol{c}_\tau = -(A^T - SB)\boldsymbol{c} \tag{22}$$

Because S is symmetric, Eq. (21) reduces to an $n(n+1)/2$-dimensional differential equation. Evaluating Eq. (18) at $\tau = 0$, the differential equation of the costate $\boldsymbol{\lambda}(t) = \boldsymbol{\lambda}^*(0, t)$ to be integrated in real time is obtained as

$$\frac{\mathrm{d}\boldsymbol{\lambda}(t)}{\mathrm{d}t} = -H_x^T[\boldsymbol{x}(t), \boldsymbol{\lambda}(t), \boldsymbol{u}(t), \boldsymbol{p}(t)] + \boldsymbol{c}(0, t) \tag{23}$$

where $\boldsymbol{x}_t^*(0, t)$ and $\boldsymbol{x}_\tau^*(0, t)$ are canceled because $\boldsymbol{x}_t^*(0, t) = \boldsymbol{x}_\tau^*(0, t)$ holds. For each time t, the Euler–Lagrange equations (7–9) are integrated forward along the τ axis, and the differential equations (21) and (22) are integrated backward with the terminal condition equations (19) and (20). Then the differential equation (23), with the initial condition equation (14), is integrated for one step along the t axis to update the costate $\boldsymbol{\lambda}(t) = \boldsymbol{\lambda}^*(0, t)$. The optimal control input is determined from Eq. (12) with the state given and the costate calculated with Eq. (23). The differential equation for the costate, Eq. (23), is different from the one in Ref. 1. If $\mathrm{d}T/\mathrm{d}t = -1$ and the optimality condition $\boldsymbol{F} = 0$ holds identically, then the third term $\boldsymbol{c}(0, t)$ vanishes and Eq. (23) is identical with the equation for the costate in the usual Euler–Lagrange equations. In fact, if $\mathrm{d}T/\mathrm{d}t = -1$ holds, i.e., $t + T$ is constant, then Eq. (2) represents a performance index at time t of a usual finite horizon optimal control problem with a fixed terminal time. Therefore, Eq. (23) gives a natural extension of the usual Euler–Lagrange equations to the case of the receding-horizon control. Also note that the matrix H_{uu} must be nonsingular, as is apparent from Eq. (16). If the matrix H_{uu} is nonsingular, the algorithm is executable regardless of controllability or stabilizability of the system. Furthermore, if H_{uu} is positive definite, the solution of the TPBVP gives a local minimum.[3]

Numerical Example

A receding-horizon tracking control law is designed for a two-wheeled car as a numerical example. The state variables of the two-wheeled car are its position (x_1, x_2) and its attitude angle x_3, and the control inputs are velocities of the wheels. The state equation of the system is given by

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = G(\boldsymbol{x})\boldsymbol{u}, \qquad G(\boldsymbol{x}) = \frac{1}{2}\begin{bmatrix} -\sin x_3 & -\sin x_3 \\ \cos x_3 & \cos x_3 \\ 1/L & -1/L \end{bmatrix} \tag{24}$$

where $\boldsymbol{x} = [x_1 \;\; x_2 \;\; x_3]^T$ denotes the state vector and $\boldsymbol{u} = [u_1 \;\; u_2]^T$ the control input vector. The distance between two wheels is denoted by $2L$. This system is controllable but is nonlinear and nonholonomic.[1] The performance index of the receding-horizon tracking control problem is chosen as the following quadratic performance index:

$$J = \frac{1}{2}\int_t^{t+T} \{[\boldsymbol{x}(t') - \boldsymbol{p}(t')]^T Q[\boldsymbol{x}(t') - \boldsymbol{p}(t')] + \boldsymbol{u}^T(t')\boldsymbol{u}(t')\}\,\mathrm{d}t' \tag{25}$$

where $Q = \mathrm{diag}(q_1, q_2, q_3) > 0$ is a weighting matrix and the reference trajectory $\boldsymbol{p}(t)$ is given by

$$\boldsymbol{p}(t) = [R_0\cos\omega_0 t \quad R_0\sin\omega_0 t \quad 0]^T \tag{26}$$

The time interval T in the performance index is given by

$$T(t) = T_f(1 - e^{-\alpha t}) \qquad (\alpha > 0) \tag{27}$$

The variable time interval $T(t)$ satisfies $T(0) = 0$, and $T(t)$ converges to the desired value T_f as the time increases. The control input is included in the performance index so that the optimal control

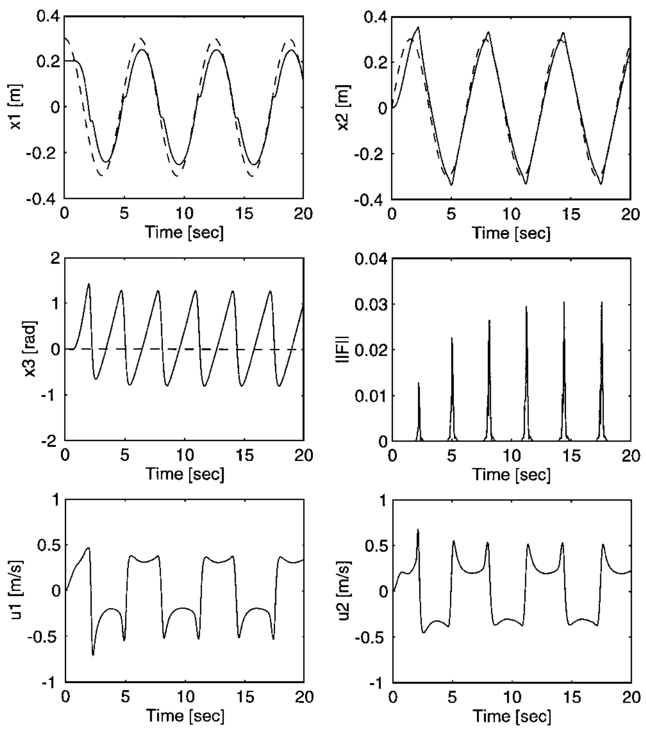

Fig. 1 Time histories of a simulation result with T_f = 1 s and Q = diag(10, 10, 0.1): – – –, reference state and ——, actual state.

has bounded magnitude. The system is controlled by the bounded control input so as to minimize the deviation from the desired reference trajectory on average over $[t, t + T]$. Because the system is nonholonomic and cannot track the reference trajectory given by Eq. (26) perfectly with any control input, the performance index is not bounded with respect to the time interval T and an infinite horizon regulator cannot be designed for the present tracking control problem. However, the receding-horizon tracking control can be designed because of the finite T.

The initial state is given in the simulation as $x_1 = 0.2$ m, $x_2 = 0$ m, and $x_3 = 0$ rad. The parameters are chosen as $L = 0.0605$ m, $R_0 = 0.3$ m, $\omega_0 = 1.0$ rad/s, and $\alpha = 0.5$. The matrix to stabilize the solution is chosen as $A_s = -50I$. The Adams method starting with the Runge–Kutta–Gill method is used for numerical integration of the differential equations on both the t axis and the τ axis. The time step on the t axis is 0.01 s, and the time step on the τ axis is 0.05 s. The simulation program is coded in C language on an Apple Power Macintosh 7100/80AV (CPU: PowerPC 601, 80 MHz; RAM: 32 MB), and the size of the binary file of the program is 66 KB. The whole computational time is about 15 s for simulating the control process of 20 s. Therefore, it can be concluded that the proposed control algorithm is implemented with a sufficiently short computational time and a moderate amount of data storage. The computational time can be reduced further by changing the integration method and/or the time steps at the expense of accuracy.

The simulation result in Fig. 1 is a case with $T_f = 1$ s and $Q =$ diag(10, 10, 0.1). Time histories of state variables; the norm of the error in the optimality condition, $\|\boldsymbol{F}\|$; and control inputs are shown in Fig. 1. The error $\|\boldsymbol{F}\|$ is less than 0.04 throughout the simulation, which validates the accuracy of the optimal solution. The peaks of $\|\boldsymbol{F}\|$ appear when the attitude angle x_3 changes rapidly. The attitude angle x_3 vibrates around the constant reference state of zero because the model cannot move in the x_1 direction with zero attitude angle. Optimization results in satisfactory tradeoff between tracking performance of x_1, x_2, and x_3. The closed-loop performance depends on free parameters of the performance index in the present optimization-based control.

Conclusions

An algorithm is proposed for the time-variant receding-horizon control of general nonlinear systems. The algorithm is derived in a manner that is different from the conventional one, and it is shown that the unknown costate is governed by a differential equation that is a natural extension of the Euler–Lagrange equations for a usual finite horizon optimal control problem with a fixed terminal time. A tracking control problem of a two-wheeled car is employed as a numerical example. It is shown that the proposed algorithm requires a realistic amount of computational time and data storage in the simulation. As the result of the numerical simulation, it is concluded that the receding-horizon tracking control achieves the best possible performance even if the reference trajectory cannot be tracked perfectly with any control input. The present approach can generate a highly nonlinear control law through optimization requiring trial and error only in selection of a few free parameters. It may be concluded that the present approach provides an efficient design technique for a wide class of nonlinear feedback control problems.

Acknowledgments

This work was partially supported by the Research Fund of the CASIO Science Promotion Foundation. The author thanks the anonymous reviewers for valuable comments.

References

[1] Ohtsuka, T., and Fujii, H. A., "Real-Time Optimization Algorithm for Nonlinear Receding-Horizon Control," *Automatica*, Vol. 33, No. 6, 1997, pp. 1147–1154.

[2] Michalska, H., "Trajectory Tracking Control Using the Receding Horizon Strategy," *Proceedings of International Workshop on Predictive and Receding Horizon Control*, Engineering Research Center for Advanced Control and Instrumentation, Seoul National Univ., Seoul, Republic of Korea, 1995, pp. 1–12.

[3] Bryson, A. E., Jr., and Ho, Y.-C., *Applied Optimal Control*, Hemisphere, New York, 1975, Secs. 2.3 and 6.3.

[4] Ohtsuka, T., "Time-Variant Receding-Horizon Control of Nonlinear Systems," AIAA Paper 96-3694, July 1996.

Intercept of Nonmoving Targets at Arbitrary Time-Varying Velocity

Ping Lu*
Iowa State University, Ames, Iowa 50011-3231

I. Introduction

PROPORTIONAL navigation (PN) guidance has been probably the most popular and extensively studied guidance method for short-range intercept. Pastrick et al.[1] provide a historical background of the PN guidance concept and a comprehensive list of the literature on PN guidance laws up to 1979. Many additional publications have since appeared on PN guidance and its variations.[2–5] In almost all of the existing studies where the capture of the target with PN guidance is analytically investigated, a key assumption is that the interceptor and the target have constant velocities. The PN guidance law in principle has no difficulty being applied to long-range intercept. For instance, in the entry flight of a lifting space vehicle, the vehicle needs to be guided from the penetration of the atmosphere to a target point near the landing site where final approach and landing maneuvers are initiated.[6] However, when the PN guidance law is applied to such a long-range intercept, the constant-velocity assumption is no longer a valid approximation because the velocity variation in this case can be as large as one order of magnitude. In this Note, we examine the PN guidance law applied to the intercept of a nonmoving target when the interceptor has arbitrary time-varying velocity. We shall show that the PN guidance law will

Received June 4, 1997; revision received July 31, 1997; accepted for publication Sept. 19, 1997.

*Associate Professor, Department of Aerospace Engineering and Engineering Mechanics. E-mail: plu@iastate.edu. Associate Fellow AIAA.